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Stability of QED Vacuum and 3 + 1 Dimensional Scattering Problem in the Presence of a Coulomb Scalar Potential and Vector Field

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Abstract We show that, in the presence of a scalar field the range of the value of external field parameters a and b, at which corresponding Hamiltonian operator is hermitian, essentially wider than in its absence. It allows us to study precisely the question on stability of QED vacuum in the presence of a strong electric field of a point charge $Z|e|$ and external scalar Coulomb field with respect to electron-positron pair production. Also, we consider the scattering of Dirac particle by the specified fields in $3+1$ dimensions. The phase shift and wave functions are obtained exactly. We calculate the scattering amplitude in a quasi-classical approximation as a partial wave series. By means of figures obtained for the cross section $\sigma(\theta)$ in general and special cases, such as $a \neq b$ and $a = b$, we find that $\sigma(\theta)$ is not exactly symmetric about $\theta = \pi$.

Keywords QED vacuum · Coulomb scalar potential and vector field · Scattering

1 Introduction

One of the important physical effects is the pair production(electron-positron) effect from vacuum by a Coulomb field [1, 2]. We remind that, the ground state of an electron in a Coulomb field of a point-charge $Z|e|$ becomes zero for $Z\alpha = 1$ ($Z = 137$), and when $Z > 137$ it becomes purely imaginary, i.e. the corresponding Hamiltonian operator becomes nonhermitian at $r = 0$. Self-adjoint modification of Hamiltonian operator is equivalent to introducing boundary condition at small r, which it means that the finite size of the nucleus should be considered [1]. The value of $Z_{cr}|e|$, at which E_0 is exactly equal to $-m$ (m- is fermion mass), is called the critical charge [3, 4]. If $Z > Z_{cr}$, then the ground state of electron is immersed in lower continuum, and if this level did not filled, from arisen quasi-stationary state two positrons are created, which under the action of Coulomb field

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will push away to infinity, and the Quantum Electrodynamics vacuum, which was perturbed by supercritical Coulomb field gets charge $2e$ [3, 4]. Really, in Dirac map, all states in lower continuum were occupied now by electrons with negative energy, so that it should not be describe the created electrons (in lower continuum) by using usual wave functions, and for their description the concept of charged vacuum must be considered [5].

In addition to interaction of fermions with an external vector field, one can consider interaction of Dirac fermions with external scalar field $U(r)$, which it models, for example the self-consistent field of quark system, or in some cases, an actual part of gravitational interaction in systems with massive fermions. Studying system of fermions in the presence of a strong vector field of point-charge and a scalar Coulomb field leads to this result that, the range of the value of external field parameters a and b, at which corresponding Hamiltonian operator becomes hermitian, essentially wider than in the absence of scalar field. Actually, it allows us to study correctly the question on stability of QED vacuum in the specified fields in relation to electron-positron pair production.

The scattering of particles by a Coulomb field is discussed in some special cases in two, three and higher dimensions in several papers [6–11]. Also, the exact solutions of the relativistic Aharonov-Bohm effect [12] in the presence of two dimensional Coulomb potential is studied in [13, 14]. It should be noted that, the study of the scattering of particles by a vector field and a Coulomb scalar potential plays an important role in relativistic quantum mechanics, because it models the interaction between the Dirac filed and heavy atoms, i.e. quarkonium.

Recently, the Dirac equation with position-dependent mass in the Coulomb field has been studied [15]. The exact solutions of the Dirac equation with the vector plus scalar Coulomb potential in two dimensions [16, 17], three dimensions [18] and higher dimensions [19] have been carried out. The pair production process in the vector fields is studied in $(1+1)$ and $(2+1)$ dimensions [20] as well as in $3+1$ dimensions [21–23].

The main purpose of this paper is to study the stability of QED vacuum in the presence of combined vector field plus Coulomb scalar potential and to obtain the scattering amplitude of Dirac particles in the presence of specified fields as a partial wave series. This article is organized as follows: In Sect. 2, we review the bound state of Dirac particles in external vector field and scalar potential. In Sect. 3, we discuss the stability of QED vacuum in the presence of these fields. Finally in Sect. 4, the phase shift and the scattering amplitude in 3-dimensions with using the quasi-classical approximation as a partial wave series are obtained.

We used the units where $c = \hbar = 1$.

2 Bound States

The Dirac equation in the presence of a vector field and a scalar Coulomb potential may be written as [24, 25],

$$[\boldsymbol{\alpha} \cdot \mathbf{p} + \beta(m + a/r) - (E - b/r)]\psi(\mathbf{x}, t) = 0, \tag{1}$$

where m is the mass of the particle, a and $b(\equiv ze^2)$ are the scalar and electrostatic coupling constants respectively. The scalar potential is added to the mass term in the Dirac equation and may be understand as an effective position-dependent mass. To separate variables we write $\psi(\mathbf{x})$ in terms of two component spinors:

$$\psi(\vec{r}) = \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} f(r)\,\Omega_{j,l,m}(\theta, \varphi) \\ (-1)^{1+l-l'} g(r)\Omega_{j,l',m}(\theta, \varphi) \end{pmatrix} \tag{2}$$

where $l = j \pm 1/2$, $l' = 2j - l$. The two component angular solutions are eigenfunctions of $\vec{J}^2$, j_z, $\vec{L}^2$ and $\vec{S}^2$ and are of two types:

$$\Omega^{(+)}_{j,l,m} = \begin{pmatrix} \sqrt{\frac{l+1/2+m}{2l+1}} Y_{l,m-1/2}(\theta,\varphi) \\ \sqrt{\frac{l+1/2-m}{2l+1}} Y_{l,m+1/2}(\theta,\varphi) \end{pmatrix}, \tag{3}$$

for $j = l + 1/2$ and

$$\Omega^{(-)}_{j,l,m} = \begin{pmatrix} -\sqrt{\frac{l+1/2-m}{2l+1}} Y_{l,m-1/2}(\theta,\varphi) \\ \sqrt{\frac{l+1/2+m}{2l+1}} Y_{l,m+1/2}(\theta,\varphi) \end{pmatrix}, \tag{4}$$

for $j = l - 1/2$, where $l = 0, 1, 2, \ldots$ and $m = -l, \ldots, l$. $Y_{l,m}(\theta,\varphi)$s' are spherical harmonics [26]. Two coupled solutions for radial equations are:

$$f(\rho) = \sqrt{m+E}\, e^{-\rho/2} \rho^{\gamma-1} (Q_1 + Q_2), \tag{5}$$

$$g(\rho) = -\sqrt{m-E}\, e^{-\rho/2} \rho^{\gamma-1} (Q_1 - Q_2), \tag{6}$$

where

$$\rho = 2\lambda r, \qquad \lambda = \sqrt{m^2 - E^2}, \qquad \gamma = \sqrt{\kappa^2 + a^2 - b^2}. \tag{7}$$

Q_1 and Q_2 are confluent hypergeometric functions, $F(\alpha, \beta; z)$;

$$Q_1 = AF\left(\gamma - \frac{bE - ma}{\lambda}, 2\gamma + 1; \rho\right), \tag{8}$$

$$Q_2 = BF\left(\gamma + 1 - \frac{bE - ma}{\lambda}, 2\gamma + 1; \rho\right), \tag{9}$$

also $\kappa = -(j + 1/2) = -(l + 1)$ for $j = l + 1/2$, and $\kappa = +(j + 1/2) = l$ for $j = l - 1/2$, and B related to A by equation:

$$B = \frac{\gamma - (bE - ma)/\lambda}{\kappa - (bE - ma)/\lambda} A. \tag{10}$$

Finally, the discrete energy levels of system are obtained as follows:

$$E_n = m\left[\frac{ab}{(n+\gamma)^2 + b^2} \pm \sqrt{\left(\frac{ab}{(n+\gamma)^2 + b^2}\right)^2 + \frac{(n+\gamma)^2 - a^2}{(n+\gamma)^2 + b^2}}\right], \tag{11}$$

where $n = 0, 1, 2, \ldots$. The ground state energy ($n = 0$ and $\gamma = \sqrt{1 + a^2 - b^2}$), is:

$$E_0 = m\left[\frac{ab + \sqrt{1 + a^2 - b^2}}{1 + a^2}\right], \tag{12}$$

which is real when $1 + a^2 > b^2 \equiv (ze^2)^2$. So we conclude that, in contrast to the pure Coulomb electric field ($a = 0$) the ground sate energy for $b = 1$ would not be zero, as shown in Fig. 1.

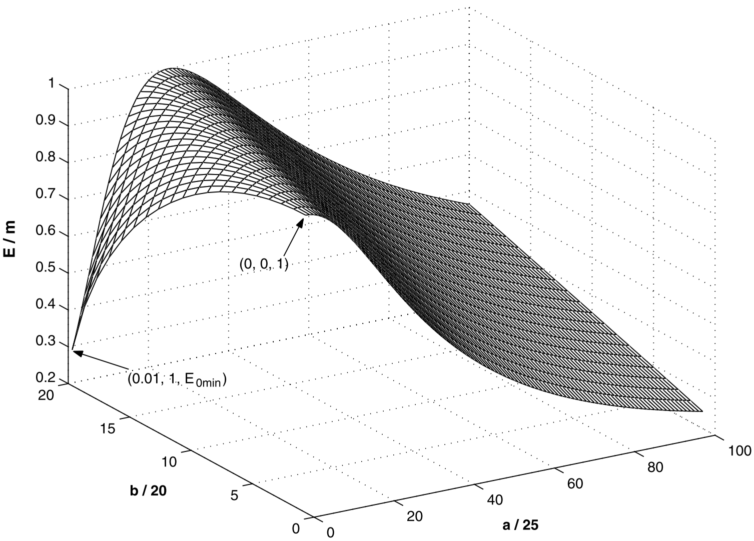

Fig. 1 The ground state energy of the fermion $E_0(a, b)$ in terms of the vector field b, and the Coulomb scalar potential a

3 Stability of QED Vacuum

The lowest energy of fermion for ground state, can be deduced from (12) if we set the boundary condition which is $1 + a^2 = b^2$

$$E_{0min} = m\frac{a}{\sqrt{1+a^2}}. \tag{13}$$

In order to study the quantum electrodynamics vacuum in the presence of a scalar Coulomb potential and vector field, we calculate the energy gap between positive (particle) and negative (anti-particle) states, by the assumption that the real electric charge is opposite with respect to the center of Coulomb vector field. The ground state energy of the anti-particle is obtained by means of exploration of (11), and for small $b < 0$ we have

$$E_0 = \frac{m}{\sqrt{1+a^2}} + m|b|. \tag{14}$$

The discrete energy levels exist for $\frac{1}{\sqrt{1+a^2}} + |b| < 1$. The energy gap in the weak scalar potential is:

$$\Delta E \cong \frac{2m}{\sqrt{1+a^2}}, \tag{15}$$

where $a^2 > 2|b|$. In the case that $b = 1$, the lowest ground state energy of the particle is zero and energy gap between particle and anti-particle becomes:

$$\Delta E \cong E_0(|b| \cong 1) = \frac{m}{\sqrt{1+a^2}} + m|b| \cong 2m, \tag{16}$$

for very large a, and

$$\Delta E \cong (m|b| + m/a) < m, \tag{17}$$

for very small a. Also it should be noted that in case of $1 + a^2 = b^2$, the lowest ground state of fermion can be obtained from (12), and if we let $a \to \infty$ then $E_0 \to -m$. Therefore the lowest energy level is unavailable for fermions, and there is no discrete level for anti-fermion for $1 + a^2 = b^2$. We conclude that, in the presence of aforementioned fields the positive energy levels for fermion never cross the negative energy levels of anti-fermions for any value of external field parameters a and b, and then, the QED vacuum will remain stable in the presence of these fields.

4 Continuous Spectrum and Scattering

The wave functions of the continuous spectrum ($E > m$) can be obtained from (5), (6) and (7) through the replacements of:

$$\sqrt{m-E} \to -i\sqrt{E-m}, \qquad \lambda \to -ip, \qquad -n \to \gamma - i\left(\frac{bE-ma}{p}\right). \tag{18}$$

Those functions should be normalized. After replacements let us represent the wave functions $f(r)$ and $g(r)$ in the form of:

$$\begin{pmatrix} f(r) \\ g(r) \end{pmatrix} = \begin{pmatrix} \sqrt{E+m} \\ i\sqrt{E-m} \end{pmatrix} A' e^{ipr}(2pr)^{\gamma-1} \times [e^{i\xi} F(\gamma - i\mu, 2\gamma+1; -2ipr) \mp e^{-i\xi} F(\gamma + 1 - i\mu, 2\gamma+1; -2ipr)], \tag{19}$$

where A' is the normalization constant and we have

$$e^{-2i\xi} = \frac{\gamma - i\mu}{\kappa - i\mu'}, \qquad \mu = \frac{bE-ma}{p}, \qquad \mu' = \frac{bm-aE}{p},$$

where ξ is real. Comparing the asymptotic behavior of (19) with the normalized spherical waves [26], A' is obtained and we have:

$$\begin{pmatrix} f(r) \\ g(r) \end{pmatrix} = 2^{3/2}\sqrt{\frac{m \pm E}{E}} \cdot \frac{(2pr)^{\gamma}}{r} \cdot \frac{|\Gamma(\gamma+1+i\mu)|}{\Gamma(2\gamma+1)} \cdot e^{\mu\pi/2} \times \begin{pmatrix} \mathrm{Im} \\ \mathrm{Re} \end{pmatrix} (e^{i(pr+\xi)} F(\gamma - i\mu, 2\gamma+1; -2ipr)), \tag{20}$$

in which $\Gamma(z)$ is the Gamma function [28].

Asymptotically, the wave function has the form:

$$\begin{pmatrix} f(r) \\ g(r) \end{pmatrix} = \sqrt{\frac{2(E \pm m)}{Er}} \begin{pmatrix} \sin \\ \cos \end{pmatrix} (pr + \delta_l + \mu Ln(2pr) - \pi l/2), \tag{21}$$

where

$$\delta_l = \xi - \pi\gamma/2 - \arg(\Gamma(\gamma+1+i\mu)) + \pi l/2, \tag{22}$$

and

$$e^{2i\delta_l} = \frac{\kappa - i\mu'}{\gamma - i\mu} \frac{\Gamma(\gamma+1-i\mu)}{\Gamma(\gamma+1+i\mu)} e^{i\pi(l-\gamma)}. \tag{23}$$

The expression for the analytical continuation of (23) in the rang $E < m$,

$$e^{2i\delta_l} = \frac{\kappa - (bm - aE)/\lambda}{\gamma - (bE - am)/\lambda} \frac{\Gamma(\gamma+1-(bE-ma))/\lambda}{\Gamma(\gamma+1+(bE-ma))/\lambda} e^{i\pi(l-\gamma)}, \tag{24}$$

has the poles at the points where $\gamma + 1 - \frac{bE-ma}{\lambda} = 1 - n, n = 1, 2, 3, \ldots$ as well at the point $\gamma - (\frac{bE-ma}{\lambda}) = -n = 0$.

In these points the energy levels are discrete. Near the poles with $n \neq 0$, it is easy to obtain,

$$e^{2i\delta_l} \approx \frac{\kappa - (bm - aE)/\lambda}{-n} \frac{\Gamma(\gamma+1-(bE-ma)/\lambda)}{\Gamma(2\gamma+1+n)} e^{i\pi(l-\gamma)}. \tag{25}$$

The residue of function in its pole is related to the coefficient in the asymptotic expression of wave function of the corresponding bound state as follows:

$$f(r) = -\sqrt{\frac{m+E}{m-E}}\, g(r) \approx \frac{A}{r} e^{-\lambda r}, \tag{26}$$

where

$$A = i^{(2\pi+\gamma)} e^{-i\gamma\pi/2} \left[\sqrt{\frac{m+E}{m-E}} \frac{(\lambda R - \kappa)\lambda}{2n!\, \Gamma(2\gamma+1+n) m R}\right]^{1/2} \cdot (2\lambda r)^{\gamma+n}, \tag{27}$$

and $R = (bm - aE)/\lambda^2$. Now using obtained relations for phase shift and wave functions for continuous spectrum, we study the scattering of fermions by external vector field and scalar Coulomb potential in 3 dimensions. The scattering amplitude for the angle of $\theta \neq 0$ is [27]:

$$f(\theta) = \sum_{l=0}^{\infty} (2l+1) P_l(cos\theta) e^{2i\delta_l}. \tag{28}$$

4.1 External Field Parameters are Equal, $a = b$

At first we study the special case where $a = b$. Substitution of the relation $a = b$ in (7) yields:

$$\gamma = \kappa,$$

then we have:

$$e^{2i\delta_l} = \frac{\Gamma(l - i\mu)}{\Gamma(l + i\mu)}. \tag{29}$$

In the quasi-classical approximation, the scattering amplitude relationship is obtained as follows:

$$f(\theta) = \frac{e^{-i\pi/4}}{k\sqrt{2\pi sin\theta}} \left\{ \sum_{l=0}^{\infty} \left[\frac{\Gamma(l-i\mu)}{\Gamma(l+i\mu)} \sqrt{l} (e^{-i(l+1/2)\theta} - i e^{+i(l+1/2)\theta}) \right] \right\}, \tag{30}$$

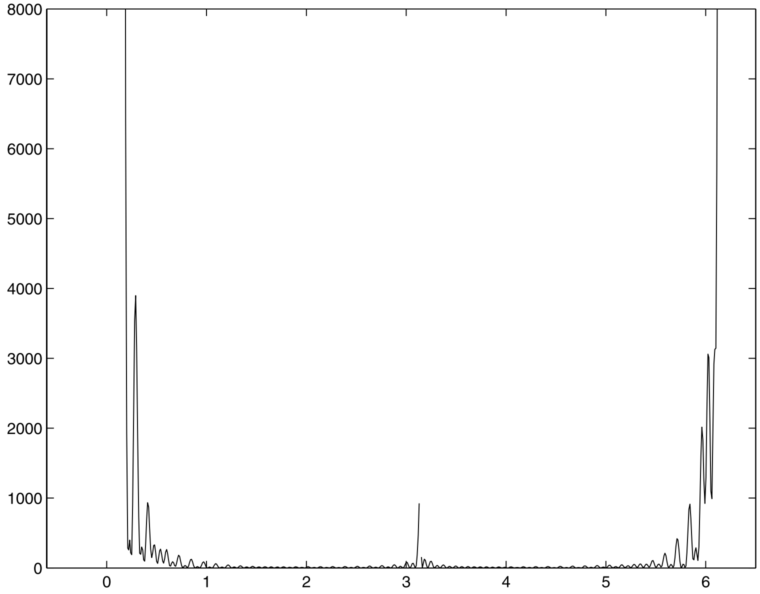

Fig. 2 The non-symmetric cross section, $\sigma(\theta)$, about $\theta=\pi$ for the case of $a=b$

from which, the cross section can be calculated as a series function and by using numerical methods it will be as Fig. 2.

$$\sigma(\theta)=|f(\theta)|^2.$$

4.2 External Field Parameters are Arbitrary, $a\neq b$

In this case, one can easily obtain the scattering amplitude

$$\begin{aligned}f(\theta)=\frac{e^{-i\pi/4}}{k\sqrt{2\pi\sin\theta}}\sum_{l=0}^{\infty}\Bigg[&\sqrt{l}\frac{l-i\mu'}{\alpha-i\mu}\frac{\alpha+1+i\mu}{\alpha+1-i\mu}\\&\times\prod_{m=1}^{\infty}\frac{m+\sqrt{l^2+a^2-b^2}+1+i\mu}{m+\alpha+1-i\mu}\\&\times(1+1/m)^{-2i\mu}\,e^{i\pi(l-\alpha)}\sin((l+1/2)\theta+\pi/4)\Bigg],\end{aligned}\tag{31}$$

where

$$\alpha=\sqrt{l^2+a^2-b^2}.$$

However, the cross section in this case can be obtained by calculating modulus of scattering amplitude equations (31) and the result for it, is shown in Fig. 3. The obtained results show that in contrast to the $2+1$ dimensional scattering problem in specified external fields [6], the cross section is not completely symmetric, neither for $a=b$ nor for $a\neq b$ about $\theta=\pi$ as it shown in Figs. 2 and 3.

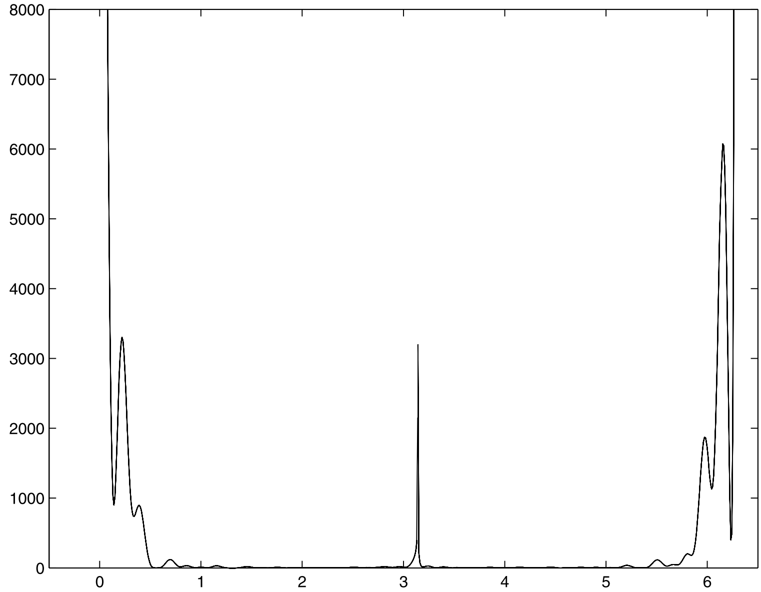

Fig. 3 The non-symmetric scattering cross section, $\sigma(\theta)$, about $\theta = \pi$ for the case of $a \neq b$. The parameters of μ, μ', κ, a and b are taken 1.7, 1.3, 1, 0.1 and 1 respectively

5 Conclusions

The study of the stability of quantum electrodynamics vacuum and $3+1$ dimensional scattering problem have shown the following:

- The QED vacuum at the presence of a vector field and a scalar Coulomb potential remain stable with respect to electron-positron pair production.
- The normalization constant of wave functions for bound state A (see (27)) is calculated by using relations for the continuous spectrum state.
- In contrast to the $2+1$ dimensional scattering problem, the cross section is not exactly symmetric about $\theta = \pi$.

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